NON-MINIMAL ELECTRODYNAMICS AND RESONANCE INTERACTIONS IN RELATIVISTIC PLASMA

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Abstract

A three-parameter toy-model, which describes a non-minimal coupling of gravity field with electromagnetic field of a relativistic two-component electrically neutral plasma, is discussed. Resonance interactions between particles and transversal waves in plasma are shown to take place due to the curvature coupling effect.

1 Introduction

In order to explain the phenomenon of accelerated expansion of the Universe and the nature of Dark Energy and Dark Matter numerous modifications of the Einstein theory of gravity are elaborated, the Non-minimal (NM) Field Theory being one of the most attractive directions in these investigations. The NM Field Theory is based on the introduction into the Lagrangian of the cross-invariants, which contain the curvature tensor in convolutions with fields of different nature and their covariant derivatives. The most detailed variants of the NM theory are elaborated for the scalar, electromagnetic and gauge fields (see, e.g., [1, 2]).

The basic principles of the NM Field Theory can also be used for the NM modification of the relativistic kinetic theory of gas and plasma [3, 4]. Since the NM electrodynamics predicts the tidal variations of the propagation velocity of the electromagnetic waves, we expect that the NM theory of relativistic plasma can exhibit new details of resonance interactions of particles and plasma waves. In particular, it is well-known that according to the minimal plasma theory the phase velocity of transversal electromagnetic waves exceeds the speed of light in vacuum, thus, there are neither resonance particle-wave interaction nor Landau damping for such waves. In the NM extension of plasma electrodynamics these phenomena are shown to be not forbidden.

In this paper we consider a model, which can be indicated as a NM modification of the well-known Vlasov theory [5]. This NM toy-model is based on a relativistic version of the Vlasov kinetic equation [5], equations of plasma electrodynamics [6] and NM extended Einstein equations [7]. For the illustration of the idea we use here a simple toy-model assuming that the plasma and electromagnetic field are the test ones, and the gravitational background is represented by the de Sitter model.

2 Master equations

(i) Relativistic kinetic equation

In the context of the Vlasov theory a 8-dimensional one-particle distribution function $f_{(a)}(x^i, p_k)$, which describes particles of a sort "(a)" with the rest mass $m_{(a)}$ and electric charge $e_{(a)}$, and is a

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function of coordinates x^i and momentum four-covector p_k, is considered to satisfy the relativistic kinetic equation of the following form [5]:

$$p_l g^{il} \left[\frac{\partial}{\partial x^i} + \Gamma^s_{ki} p_s \frac{\partial}{\partial p_k} + \frac{e_{(a)}}{c} F_{ki} \frac{\partial}{\partial p_k} \right] f_{(a)} = 0 \,. \tag{1}$$

Here Γ^s_{ki} are the Christoffel symbols associated with the space-time metric g_{ik}. The Maxwell tensor F_{ki} describes a macroscopic averaged self-consistent electromagnetic field in plasma, which guides the dynamics of charged particles.

(ii) Non-minimal electrodynamic equations

The master equations for the electromagnetic field have the standard form:

$$\nabla_k H_i^{\ k} = -4\pi \sum_{(a)} e_{(a)} N^{(a)}_i \,, \tag{2}$$

$$\nabla_i F_{kl} + \nabla_l F_{ik} + \nabla_k F_{li} = 0 \,, \tag{3}$$

where ∇_k is the covariant derivative. According to the Vlasov approach the particle number four-covector $N^{(a)}_i$ can be represented by the following integral

$$N^{(a)}_i = \int \frac{d_4 P}{\sqrt{-g}} \, p_i f_{(a)} \delta \left[\sqrt{g^{ik} p_i p_k} - m_{(a)} c \right] \theta(V^k p_k) \,. \tag{4}$$

Here $d_4 P \equiv dp_0 dp_1 dp_2 dp_3$ symbolizes the four-volume in the momentum space; the delta function guarantees the normalization property of the particle momentum, i.e., $g^{ik} p_i p_k = m^2_{(a)} c^2$; the Heaviside function $\theta(V^k p_k)$ rejects negative energy, V^k is a velocity four-vector of the system as a whole. According to the NM approach [1, 2, 7], the simplest form of the electromagnetic excitation tensor H^{ik} can be written as follows

$$H^{ik} = F^{ik} + \mathcal{R}^{ik}_{\ \ mn} \, F^{mn} \,, \tag{5}$$

where the NM susceptibility tensor

$$\mathcal{R}^{ikmn} = \frac{q_1}{2} R \, (g^{im} g^{kn} - g^{in} g^{km}) +$$

$$\frac{q_2}{2} (R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in}) + q_3 R^{ikmn} \tag{6}$$

contains three arbitrary parameters of NM coupling q_1, q_2 and q_3 with a dimensionality of length in square.

(iii) Gravitational background

The NM three-parameter extension of the Einstein equations was discussed, e.g., in [7]. When the plasma and its electromagnetic field can be regarded as the test ones, one can use the concept of background gravitational field. Below we follow this concept and consider the de Sitter metric

$$ds^2 = a^2(\eta) \left[d\eta^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \right] \tag{7}$$

with $a(\eta) = \frac{a_0}{\eta}$ as a gravitational background for the NM plasma electrodynamics. For this metric the formulas for the NM susceptibility (6) yield

$$\mathcal{R}^{ik}_{\ \ mn} = \mathcal{K} \, (\delta^i_m \delta^k_n - \delta^i_n \delta^k_m) \,, \tag{8}$$

where the constant $\mathcal{K}$, defined as

$$\mathcal{K} \equiv -\frac{1}{a_0^2} \, (6q_1 + 3q_2 + q_3) \,, \tag{9}$$

may be positive, negative or equal to zero depending on the values of NM parameters q_1, q_2 and q_3.

3 Non-minimally coupled electromagnetic waves in a plasma

Stationary 7-dimensional distribution function, the solution of (1), which guarantees that the macroscopic averaged collective electromagnetic field in the electrically neutral plasma vanishes, is well-known [8]

$$f^{(\rm st)}_{(a)} = f^{(0)}_{(a)}(q^2) \,, \quad q^2 \equiv p_1^2{+}p_2^2{+}p_3^2 \,, \tag{10}$$

where $f^{(0)}_{(a)}(q^2)$ is arbitrary function of its argument. The component p_0 of the particle momentum

$$p_0 = a^2 p^0 = \sqrt{m^2_{(a)} c^2 a^2(\eta) + q^2} \,, \tag{11}$$

coincides with q in the ultrarelativistic limit. The perturbed quantities: distribution function $\delta f_{(a)}$ and the Maxwell tensor $\delta F_{ik} \equiv F_{ik}$, satisfy in our toy-model to the system of integro-differential equations:

$$\left[q \ \frac{\partial}{\partial \eta} + q^{\gamma} \frac{\partial}{\partial x^{\gamma}}\right] \delta f_{(a)} = \frac{e_{(a)}}{c} \ F_{\gamma 0} \ q^{\gamma} \ \frac{\partial f^{(0)}_{(a)}}{\partial q} \,, \tag{12}$$

$$(1{+}2\mathcal{K}) \frac{\partial}{\partial x^{\alpha}} F_{0\alpha} = 4\pi \sum_{(a)} e_{(a)} \int d^3 q \ \delta f_{(a)} \,, \tag{13}$$

$$(1+2\mathcal{K}) \left[\frac{\partial}{\partial x^{\alpha}} F_{\gamma\alpha} - \frac{\partial}{\partial \eta} F_{\gamma 0}\right] =$$

$$= -4\pi \sum_{(a)} e_{(a)} \int \frac{q^{\gamma}}{q} d^3 q \ \delta f_{(a)} \,, \tag{14}$$

$$\frac{\partial F_{\gamma\alpha}}{\partial \eta} + \frac{\partial F_{\alpha 0}}{\partial x^{\gamma}} + \frac{\partial F_{0\gamma}}{\partial x^{\alpha}} = 0 \,. \tag{15}$$

Here we introduced convenient notations

$$q^{\alpha} \equiv a^2 g^{\alpha\gamma} p_{\gamma} \,, \quad q^2{=}(q^1)^2{+}(q^2)^2{+}(q^3)^2 \,, \tag{16}$$

and $d^3 q \equiv dq^1 dq^2 dq^3$, the summation over the repeating indices being assumed. The non-minimal equations (12) - (15) form a system of linear integro-differential equations with coefficients, which do not depend on time, and it is a good surprise. This system can be resolved using the Fourier transformation

$$\mathbf{H}(\eta, x^{\alpha}){=}\int_{-\infty}^{\infty} \frac{d\omega d_3 k}{(2\pi)^4} \mathcal{H}(\omega, k_{\alpha}) e^{i\left(k_{\alpha} x^{\alpha} - \frac{\omega\eta}{c}\right)} \,, \tag{17}$$

where calligraphic letter $\mathcal{H}(\omega, k_{\alpha})$ denotes the Fourier transform of the function $\mathbf{H}(\eta, x^{\alpha})$; $d_3 k \equiv dk_1 dk_2 dk_3$. Simple calculations give

$$\delta f_{(a)}(\omega, k_{\alpha}){=}\mathcal{F}_{\gamma 0}(\omega, k_{\alpha}) \left\{ \frac{i e_{(a)} q^{\gamma}}{q\left(\omega{-}\frac{c k_{\alpha} q^{\alpha}}{q}\right)} \ \frac{\partial f^{(0)}_{(a)}}{\partial q} \right\} \,, \tag{18}$$

$$\mathcal{F}_{\alpha 0} \left[(1{+}2\mathcal{K}) \frac{k^2 c^2}{\omega^2} \left(\frac{k_{\alpha} k_{\beta}}{k^2} {-} \delta_{\alpha\beta}\right) {+} 2\mathcal{K} \delta_{\alpha\beta} {+} \varepsilon_{\alpha\beta}\right] {=} 0 \,, \tag{19}$$

where $\varepsilon_{\alpha\beta}$ denotes a permittivity three-tensor

$$\varepsilon_{\alpha\beta} \equiv \delta_{\alpha\beta}+\frac{4\pi}{\omega}\sum_{(a)} e^2_{(a)} \int \frac{d^3q\ q^{\alpha}q^{\beta}}{q^2\left(\omega-\frac{ck_{\gamma}q^{\gamma}}{q}\right)}\ \frac{\partial f^{(0)}_{(a)}}{\partial q}\,. \tag{20}$$

Since the zero-order distribution function $f^{(0)}_{(a)}$ is considered to depend on q^2, the permittivity tensor has an isotropic structure

$$\varepsilon_{\alpha\beta} = \left(\delta_{\alpha\beta} - \frac{k_{\alpha}k_{\beta}}{k^2}\right)\varepsilon^{(\mathrm{tr})} + \frac{k_{\alpha}k_{\beta}}{k^2}\varepsilon^{(\mathrm{l})}\,, \tag{21}$$

thus, the dispersion relations for the longitudinal and transversal electromagnetic waves are, respectively,

$$1-\varepsilon^{(\mathrm{l})}(\omega,k_{\alpha}) = 1+2\mathcal{K}\,, \tag{22}$$

$$1-\varepsilon^{(\mathrm{tr})}(\omega,k_{\alpha}) = (1+2\mathcal{K})\left(1-\frac{k^2c^2}{\omega^2}\right)\,. \tag{23}$$

When $\mathcal{K}=0$, the equations (22) and (23) coincide with the well-known dispersion relations (see, e.g., [6]), as it should be. Since the scalar permittivities $\varepsilon^{(\mathrm{l})}$ and $\varepsilon^{(\mathrm{tr})}$ do not depend on the NM parameters, they coincide with the ones, obtained by Silin in the framework of minimal theory of ultrarelativistic plasma [9].

The dispersion relations for longitudinal plasma waves (see (22)) and for transversal electromagnetic waves (23) depend essentially on the sign of the parameter $1{+}2\mathcal{K}$ with $\mathcal{K}$ given by (9). This parameter is predetermined both by the space-time curvature and by constants of non-minimal coupling. Below we consider all three cases, when this parameter is positive, negative or vanishes.

(I) *First case:* $1{+}2\mathcal{K}>0$

Using the NM reparametrization of the Debye radius $r_{\mathrm{D}} \to r^*_{\mathrm{D}} \equiv r_{\mathrm{D}}\ \sqrt{1{+}2\mathcal{K}}$, where

$$\frac{1}{r^2_{\mathrm{D}}} \equiv \sum_{(a)} \frac{4\pi e^2_{(a)}N_{(a)}}{k_B T_{(a)}}\,, \tag{24}$$

(see [6, 9]), one can show explicitly, that all the results for the longitudinal and transversal waves [6, 9] in the ultrarelativistic plasma remain (qualitatively) valid.

(II) *Second case:* $1{+}2\mathcal{K}=0$

Such a value of the parameter $\mathcal{K}$ relates to the case of vanishing excitation tensor H^{ik}. The Maxwell equations (2) remain consistent if the electric current vanishes. This means that self-consistent averaged electromagnetic field should also vanish (see (13), (14) and (18)), thus, there are no plasma waves in such a system.

(III) *Third case:*$1{+}2\mathcal{K}<0$

This case gives principally new results. After the NM reparametrization of the Debye radius $r_{\mathrm{D}} \to r^*_{\mathrm{D}} \equiv r_{\mathrm{D}}\ \sqrt{|1{+}2\mathcal{K}|}$ in terms of dimensionless variables

$$z = x+iy \equiv \frac{\omega}{kc} = \frac{\Omega}{kc} + i\frac{\gamma}{kc}\,, \tag{25}$$

where $\Omega(k)$ is a frequency of plasma oscillations, and $\gamma(k)$ is a decrement of damping or increment of instability (depending on its sign), the dispersion relation for *longitudinal* plasma waves takes the form

$$2\left(k^2r^{*2}_{\mathrm{D}}-1\right) = \Phi(z)\,, \tag{26}$$

where

$$\Phi(z) \equiv z \log \frac{|z-1|}{|z+1|} + i\pi z \Theta\left(1 - \Re e z\right) . \tag{27}$$

In the minimal plasma electrodynamics there exists a solution with $k = 0$ and $\omega = c/\sqrt{3} r_{\rm D}$, but the solution with $\omega = 0$ for real wave numbers k does not exist. The situation in the NM plasma electrodynamics can be just opposite: the equation (26) admits the solution $\omega = 0$, when the wave length is equal to the modified Debye radius, i.e., $k r^*_{\rm D} = 1$. As for a solution with $k \to 0$, the equation (26) gives $\omega^2 + c^2/3 r^{*2}_{\rm D} = 0$, i.e., there are no oscillations, the process of plasma evolution is aperiodic.

When we deal with *transversal* electromagnetic waves, the dispersion relation reduces to

$$\frac{2z^2}{1-z^2} - 4k^2 r^{*2}_{\rm D} = \Phi(z) \,. \tag{28}$$

Numerical analysis of the solution $z(k)$ of this equation (we omit it in this short report) shows that there exist solutions with $\Re e z < 1$. This principally new result can be illustrated qualitatively in the case, when the imaginary part of the solution $z(k)$ is much smaller than the real one, i.e., $|\gamma| << |\Omega|$. Then the real part $x = \Re e z$ should be found from the real equation

$$4k^2 r^{*2}_{\rm D} = \frac{2x^2}{1-x^2} - x \log \frac{|x-1|}{|x+1|} \equiv \Psi(x) \,. \tag{29}$$

Clearly, the function $\Psi(x)$ increases monotonically from zero at $x = 0$ to infinity, when $x \to 1_{-0}$, remaining positive for arbitrary x from the interval $0 < x < 1$. Thus, for each value of positive parameter $4k^2 r^{*2}_{\rm D}$ one can find the corresponding point on the plot of $\Psi(x)$, so, the solution $x = X(k)$ exists for all k. Analogously, it is easy to check, that for the interval $x > 1$ there are no solutions. Since the dispersion relation admits the solution with the phase velocity less that speed of light in vacuum ($\Omega/k < c$), the imaginary part of frequency, γ, is non-vanishing:

$$\gamma = \frac{1}{4} \pi k c \, \frac{\left(\frac{\Omega}{kc}\right)^2 \left[1 - \left(\frac{\Omega}{kc}\right)^2\right]^2}{\left(\frac{\Omega}{kc}\right)^2 + \left[1 - \left(\frac{\Omega}{kc}\right)^2\right]^2 k^2 r^{*2}_{\rm D}} \,. \tag{30}$$

When $1{+}2\mathcal{K}$ is negative, the parameter γ becomes positive, i.e., it describes an increment of instability. The parameter γ disappears, when $\Omega{=}0$ and when $\Omega{=}kc$, thus, being positive, it reaches a maximum on the interval $0 < \Omega < kc$. For small values of Ω/kc the condition $\gamma << \Omega$ is valid, when $k^2 r^{*2}_{\rm D} >> 1$; for $\Omega \simeq kc$, it is valid for arbitrary value of the parameter $k r^*_{\rm D}$.

4 Conclusions

1. The discussed non-minimal toy-model demonstrates an explicit example that the effect of curvature coupling provides transversal electromagnetic waves to propagate in a relativistic plasma with a phase velocity less than speed of light in vacuum.
2. Curvature coupling can support resonance interactions of transverse electromagnetic waves with plasma particles.
3. In case of domination of the non-minimal interactions ($2\mathcal{K} < -1$) the resonance interactions of plasma particles with transversal electromagnetic waves produce an instability, which is an antipode of Landau damping.

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